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LETTER TO THE EDITOR

Quantum electrodynamics: large N_f computation of the critical exponent η in arbitrary dimensions

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Abstract. We compute the critical exponent η, which relates to the anomalous dimension of the electron, in the large N_f expansion of QED at leading order in arbitrary dimensions in the Landau gauge. Subsequently we show that transcendental numbers will arise at fifth order and beyond in the corresponding renormalization group function in four dimensions.

The quantum properties of a renormalized quantum field theory are characterized by the renormalization group equation which depends on the renormalization group functions, such as the β-function and the anomalous dimension, $\gamma(g)$. The latter are ordinarily calculated in perturbation theory by rendering various Green functions of the theory finite to a particular order. Invariably, though, one can only calculate to a few orders in perturbation theory as the integrals which arise become intractable. To probe perturbation theory more deeply, alternative techniques are required. For low dimensional σ models, such methods are available [1, 2]. There one solves the model at the d-dimensional critical point, $2 < d < 4$, within the large N expansion, by computing various critical exponents in arbitrary dimensions, which are related to the critical renormalization group functions. Hence, one can deduce information to all orders in perturbation theory at the appropriate order of the large N expansion.

As quantum electrodynamics (QED) admits a large N_f expansion, where N_f is the number of electron flavours, it is the purpose of this letter to apply the methods of [2] to deduce the exponent η at $O(1/N_f)$ in arbitrary dimensions. The theory is similar to the σ models in that it has a coupling constant which is dimensionless in a particular dimension, in this case four, and hence there is a non-trivial critical point at g_c, where $\beta(g_c) = 0$ in $d < 4$. This is of interest for various reasons. As the exponent is determined in arbitrary dimensions, we will be able to gain information concerning the perturbative structure of $\gamma(g)$ to all orders in perturbation theory, as well as to gain η in three dimensions, where QED is super-renormalizable. Various authors have examined three-dimensional QED to understand mass generation as well as the thermal properties of gauge theories [3, 4] and other problems [5]. Secondly, the current situation with the large N_f expansion for QED is that the model has only been solved analytically to $O(1/N_f)$ [3]. As the method of [2] provides a straightforward way of going beyond this order, which has been carried out in a number of other models [6–8], it is important to draw attention to its potential application to QED. We note that a previous application of the theory of critical behaviour to gauge theories included solving for the infrared asymptotic properties of the gluon propagator in the absence of matter fields [9].

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We begin by recalling the important features of QED required here and to fix notation we note that the massless Lagrangian is

$$L = i\bar{\psi}^i \partial\!\!\!/ \psi^i + A_\mu \bar{\psi}^i \gamma^\mu \psi^i - \frac{(F_{\mu\nu})^2}{4e^2} - \frac{(\partial_\mu A^\mu)^2}{2be^2} \tag{1}$$

where e is the electron coupling constant, and appears in the kinetic term of the U(1) gauge field, A_μ, in anticipation of the application of the exponent method [2], $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and ψ^i, with $1 \leq i \leq N_f$, correspond to the N_f electron flavours. We have also included the covariant gauge fixing term, with gauge parameter b. The β-function for (1) in minimal subtraction is [10–12]

$$\beta(g) = (d-4)g + \tfrac{2}{3}N_f g^2 + \tfrac{1}{2}N_f g^3 - \frac{N_f}{144}(22N_f + 9)g^4 + O(g^5) \tag{2}$$

where we use the conventions of [12] but have defined $g = (e/2\pi)^2$, and (2) defines the non-trivial zero, g_c. The anomalous dimension of the fermions is [11],

$$\gamma(g) = \tfrac{1}{2}bg - \tfrac{1}{16}(4N_f + 3)g^2 + O(g^3) \tag{3}$$

in the same scheme.

The method we use, based on [2], involves solving the skeleton Dyson equations with dressed propagators for the fields ψ^i and A_μ precisely at the non-trivial critical point. As in [2] we assume the fields satisfy asymptotic scaling and define the respective asymptotic scaling functions in momentum space as [6–9]

$$\psi(k) = \frac{Ak\!\!\!/}{(k^2)^{\mu-\alpha}} \tag{4}$$

$$A_{\nu\sigma}(k) = \frac{B}{(k^2)^{\mu-\beta}}\left[\eta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2}\right] - bB\frac{k_\nu k_\sigma}{(k^2)^\mu}. \tag{5}$$

The quantities α and β are the critical exponents of each field and are functions of $d = 2\mu$, the dimension of spacetime, and N_f by the universality principle. Also, A and B are the amplitudes of each field and do not depend on the momentum k or the exponents [2]. Ensuring that these propagators satisfy the Dyson equations leads to the consistency equations which give the exponents. In (5), it is clear that the transverse and longitudinal pieces of the full propagator are of differing dimensions [6, 9], which is because the longitudinal part is not affected by the interaction [13]. Thus the total propagator in a general covariant gauge, $b \neq 0$, is not conformally invariant. However, in this letter we consider only the Landau gauge, $b = 0$, where the propagator is conformal, and focus on the transverse part of the propagator, as the method does not appear to be applicable in the $b \neq 0$ case [9]. The exponents of the fields can be related to the more conventional exponents via

$$\alpha = \mu - 1 + \tfrac{1}{2}\eta \qquad \beta = 1 - \eta - \chi \tag{6}$$

where η and χ are, respectively, the anomalous dimension of the electron field and the electron photon interaction of (1), and both are $O(1/N_f)$ in the large N_f expansion.

The skeleton Dyson equations satisfied by each field are given in figure 1 where we ignore tadpole graphs as they do not contribute to the scaling. The inverses of (4) and (5) are denoted by ψ^{-1} and A^{-1} respectively where, in keeping with [6, 9], we invert $A_{\mu\nu}$ on the transverse subspace, and thus

$$\psi^{-1} = \frac{k\!\!\!/}{A(k^2)^{\alpha-\mu+1}} \qquad A^{-1}_{\nu\sigma} = \frac{1}{B(k^2)^{\beta-\mu}}\left[\eta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2}\right]. \tag{7}$$

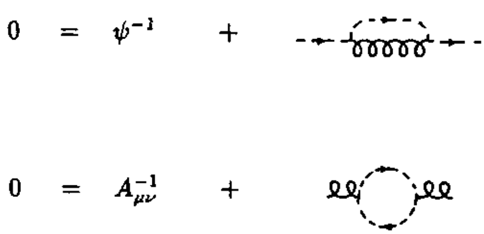

Figure 1. Skeleton Dyson equations with dressed propagators.

With (4), (5) and (7), it is straightforward to compute the (massless) one loop integrals of figure 1 to obtain

$$0=1+\frac{\alpha(2\mu-1)(\beta+1-\mu)}{(\mu-\beta)(\alpha+\beta)}\,\nu(\mu-\beta,\mu-\alpha,\alpha+\beta)z \tag{8}$$

from the first graph, where we have used the rules for integrating chains of propagators [7, 14]. We have set $z=A^2B$ and defined $\nu(\alpha_1,\alpha_2,\alpha_3)=\pi^\mu\prod_{i=1}^3 a(\alpha_i)$, where $a(\alpha)=\Gamma(\mu-\alpha)/\Gamma(\alpha)$. The powers of p^2, where p is the non-zero momentum, have cancelled since $\chi=0$ in this leading order approximation [2]. For the second graph, only the transverse part is relevant and projecting out, one is left with

$$0=1+\frac{8\alpha^2N_{\mathrm{f}}}{(\mu-\alpha-1)(2\alpha+1)}\,\nu(\mu-\alpha-1,\mu-\alpha,2\alpha+1)z. \tag{9}$$

Eliminating z from (8) and (9) gives the consistency equation satisfied by η, which is the only unknown, and with $\alpha=\mu-1+\frac{1}{2}\eta$, $\beta=1-\eta$ and $\eta=\Sigma_{i=1}^{\infty}\eta_i/N_{\mathrm{f}}^i$, we obtain

$$\eta_1(\mu)=\frac{(2\mu-1)(2-\mu)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu+1)\Gamma(2-\mu)}. \tag{10}$$

We make several remarks concerning (10), which is the main result of this letter. Fist, setting $d=4-2\varepsilon$ in (10) and expanding in powers of ε one finds agreement with the two loop anomalous dimension $\gamma(g)$, (3), in the Landau gauge since they are related via $\eta=\gamma(g_{\mathrm{c}})$. Moreover, one can go beyond the two-loop expression of (3) and deduce the coefficients of the leading order terms in $\gamma(g)$ to all orders in perturbation theory. First, from (2),

$$g_{\mathrm{c}}=\frac{3}{N_{\mathrm{f}}}\,\varepsilon-\frac{27}{4N_{\mathrm{f}}^2}\,\varepsilon^2+\mathrm{O}\left(\frac{\varepsilon^3}{N_{\mathrm{f}}^2}\right) \tag{11}$$

so that the first term of (11) is relevant for the higher order terms of (3). Setting

$$\gamma(g)=-\tfrac{1}{16}(4N_{\mathrm{f}}+3)g^2+\sum_{n=2}^{\infty}a_nN_{\mathrm{f}}^ng^{n+1}+\mathrm{O}\left(\frac{1}{N_{\mathrm{f}}^2}\right) \tag{12}$$

where we recall that gN_{f} is held fixed in the large N_{f} expansion, then the unknown coefficients, a_n, can be read off from (10). In particular, the first few terms give

$$a_n=\tfrac{5}{72}\qquad a_3=\tfrac{35}{1296}\qquad a_4=\tfrac{1}{54}\left(\tfrac{1471}{1728}-\zeta(3)\right) \tag{13}$$

which we believe have not been given before. Hence, a transcendental coefficient will arise at fifth order in perturbation theory and more generally $\zeta(n)$ will appear at $\mathrm{O}(g^{n+2})$, $n\geqslant 3$. Second, we can evaluate η in three dimensions and find

$$\eta_1(\tfrac{3}{2})=-\frac{8}{3\pi^2} \tag{14}$$

which is in agreement with recent calculations [4, 5], though these were carried out in strictly three dimensions, which provides another check on our result.

We conclude noting that we have demonstrated that the method of [2] can be applied to QED to obtain results consistent with the known results of perturbation theory and to deduce the structure at higher orders. To go beyond the order computed here, one has not only to expand (8) and (9) to $O(1/N_f^2)$ but also to include the two- and three-loop graphs which contribute at the next order, which will give a deeper insight into the nature of both the three- and four-dimensional models.

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